FUNCTIONAL ANALYSIS FORMULAS

NORM DEFINITION

Norm Definition: $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$

Scalar Rule: $\|\alpha x\| = |\alpha| \|x\|$

Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$

Normed Space: $(X, \|\cdot\|)$

Banach Space: Complete normed space

Inner Product: $\langle x, y \rangle$

INDUCED NORM

Induced Norm: $\|x\| = \sqrt{\langle x, x \rangle}$

Cauchy-Schwarz: $|\langle x, y \rangle| \leq \|x\| \|y\|$

Parallelogram Law:
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Hilbert Space: Complete inner product space

Orthogonality: $\langle x, y \rangle = 0$

Orthonormal Set: $\langle e_i, e_j \rangle = \delta_{ij}$

PROJECTION

Projection: $\mathrm{proj}_y x = \frac{\langle x, y \rangle}{\|y\|^2} \cdot y$

Gram-Schmidt: Orthonormalization

Bessel Inequality: $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq \|x\|^2$

Parseval Identity: $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 = \|x\|^2$

Linear Operator: $T(x+y) = T(x) + T(y)$, $T(\alpha x) = \alpha T(x)$

Bounded Operator: $\|Tx\| \leq C\|x\|$

OPERATOR NORM

Operator Norm: $\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$

Adjoint Operator: $\langle Tx, y \rangle = \langle x, T^* y \rangle$

Self-Adjoint: $T = T^*$

Unitary Operator: $T^* T = I$

Compact Operator: Maps bounded → compact

Spectrum: $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$

SPECTRAL RADIUS

Spectral Radius: $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$

Hahn-Banach: Extend functionals

Open Mapping: Onto linear map is open

Closed Graph: Closed graph ⇒ bounded

Uniform Boundedness: $\sup \|T_n x\|$ bounded

Weak Convergence: $\langle x_n, f \rangle \to \langle x, f \rangle$

WEAK* CONVERGENCE

Weak* Convergence: Dual space convergence

Reflexive Space: $X \cong X^{**}$

L^p Space: $\|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{1/p}$

L^1 Norm: $\|f\|_1 = \int_\Omega |f(x)| \, d\mu$

L^2 Norm: $\|f\|_2 = \left(\int_\Omega |f(x)|^2 \, d\mu \right)^{1/2}$

L^∞ Norm: $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \Omega} |f(x)|$

HOLDER INEQUALITY

Holder Inequality: $\int_\Omega |fg| \, d\mu \leq \|f\|_p \|g\|_q$

Minkowski Inequality:
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Banach Fixed Point: Contraction ⇒ unique fixed point

Sobolev Space: Functions with weak derivatives